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A COMPARISON OF PROBABILISTIC TECHNIQUES FOR THE STRENGTH OF FIBROUS MATERIALS UNDER LOCAL LOAD-SHARING AMONG FIBERS†

S. L. Phoenix

Sibley School of Mechanical, and Aerospace Engineering, Cornell University, Ithaca, NY 14853, U.S.A.

and

R. L. Smith

Department of Mathematics, Imperial College of Science and Technology, London SW7 2BZ, England

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Abstract—In this paper we compare three distinct asymptotic analyses for the probability distribution for strength of a certain class of fibrous materials. This class, which contains composite materials and twisted yarns and cables, is characterized by strong, mechanical interaction among neighboring fibers as is generated by a binding matrix or by transverse frictional forces. The model is the chain-of-bundles model, and we focus mainly on a planar geometry and a simple type of local load-sharing among fibers in a cross-section wherein the load of a failed fiber is shifted in equal portions to its two nearest flanking survivors. Through the study of typical examples we find that the results of the three methods all underscore the importance of the Weibull distribution for modeling composite strength, and are in excellent numerical agreement despite their differing analytical form. This is fortunate because only the two least accurate of the analyses show promise of being extended to more general three-dimensional geometries and local load-sharing rules of interest in applications. The paper lays the groundwork for considering such extension in future papers.

1. INTRODUCTION

In this paper we consider the strength of fibrous materials which have strong, mechanical interaction between neighboring fibers as would be generated by a binding matrix or friction. Composite materials, and twisted yarns and cables are examples. Randomly occurring flaws cause the fibers to have highly variable strength so that a complicated mechanism of local load redistribution around fiber breaks becomes involved in determining the overall material strength. Indeed the strength of such materials does not follow a simple "rule of mixtures."

In the past, various attempts have been made to develop micromechanical models of the statistical failure process. Almost exclusively the model considered has been the *chain-of-bundles model* under various versions of *local load-sharing* wherein load concentration factors are prescribed for fibers adjacent to broken fibers. We retain this model here. Zweben[1], Zweben and Rosen[2], Scop and Argon[3] and Argon[4] all considered versions of this model, and derived some interesting approximate results. Unfortunately, the model, while easily described, has proven very difficult to analyze, and exact results have been possible only for a few special cases involving very few fibers[5, 6]. Present computer capabilities do not permit extending these exact analyses to the typical case where the number of fibers is large.

Recently, attention has turned to the development of asymptotic methods, and three distinct asymptotic analyses have emerged. The first of these methods, developed by Harlow and Phoenix[7–10] is a recursion analysis for sequences of bundles of increasing size, and considers the event that k adjacent fiber breaks (k-failure) occur somewhere in the material. The notion is that if k is chosen sufficiently large, k-failure is equivalent to total failure.

The recursive technique has yielded a simple mathematical structure for the probability distribution of composite strength, and is extremely accurate. Unfortunately the most important functions that arise must be calculated numerically, a process which rapidly becomes tedious and expensive as k increases. Furthermore the method has been applied successfully only to the simplest geometries and load-sharing rules; unrestricted extension to more general settings has so far proved impractical[11, 12]. Nevertheless the recursive technique has served as a "benchmark" for evaluating other asymptotic techniques, and has yielded great insight.

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The *second* of the asymptotic analyses, developed by Smith[13, 14], also considers the event of k-failure in the fibrous material. Certain reasonable assumptions are made about the lower tail of the distribution function for fiber strength, and a limiting Weibull distribution is obtained for material strength. The method shows promise of being applicable to general geometries and types of local load-sharing, and yields results which have an appealing form.

Unfortunately this second approach has serious drawbacks. First, the results are sensitive to the numerical choice for k for which only rough procedures for its determination are available. Second, the use of certain inequalities in the analysis precludes the direct assessment of the errors in the asymptotic approximations. Third, certain key constants become difficult to express and calculate as k increases.

To overcome the above shortcomings Smith[13, 15] has developed a *third* asymptotic analysis which again considers the event of k-failure but under the assumption that the variability in fiber strength is small. It emerges that a certain value of k becomes critical, that is, a sequence of adjacent fiber breaks once reaching this length, suddenly grows catastrophically. With this approach, some simple approximating distributions for strength are obtained.

The main advantages of this third analysis are that a straightforward method is obtained for evaluating the critical k, the key constants are easily expressed, and the approach shows promise of being extended to more general settings. The major weakness is that the accuracy of the resulting approximations cannot be evaluated directly.

The purpose of this paper is to compare in depth the three asymptotic analyses and their results. We will pay special attention to the impact of certain assumptions, to the sources of error in the resulting approximations, and to the consistency of the final predictions in typical material settings. We will work with a simple planar geometry and a simple type of local load-sharing. In typical material settings the numerical predictions of the three methods will turn out to be in very close agreement, and many of the key quantities in each analysis will turn out to be intimately related. By considering all three analyses simultaneously, we will be able to determine the most useful of the various approximations, and we will appreciate why these approximations perform as well as they do in situations important in applications. The groundwork will then be laid for extending the various approximations to more general and realistic geometries and fiber load-sharing settings. Experimental verification of the predictions of the model is being carried out, and will be reported on elsewhere.

In Section 2, we describe the model and basic assumptions. Section 3 discusses the main results of the recursion analysis; this will be called Analysis I. In Section 4 we consider a motivating example under assumptions for fiber strength which span the range of practical interest. Sections 5 and 6 describe the two asymptotic analyses which will be referred to respectively as Analyses II and III. Important technical ideas in the analysis are given in the Appendix. In Section 7 we give numerical results which bring out clearly the relations among the three analyses.

2. THE MODEL AND REVIEW OF EARLIER ANALYSIS

The fibrous material is viewed as a planar structure of n parallel fibers, and this structure is conceptually partitioned into a series of m short sections called bundles, each with n fiber elements as shown in Fig. 1. The length of each bundle is the "ineffective length" δ and the material length is thus $l = m\delta$. The bundles are assumed to be statistically independent, and the strength of the fibrous material is that of its weakest bundle. We will measure the strength on a load per fiber basis; that is, the strength of the structure is the total load it can support divided by the number of fibers n.

Assumptions on fiber strength

We assume that the strengths of the mn fiber elements are independent and identically distributed random variables with common distribution function which we denote by $F(x)$, $x \geqq 0$. Two particular versions of $F(x)$ will be considered. The first is the *Weibull distribution function*

$$F(x) = 1 - \exp\{-(x/x_\delta)^\rho\}, \quad x \geqq 0 \tag{2.1}$$

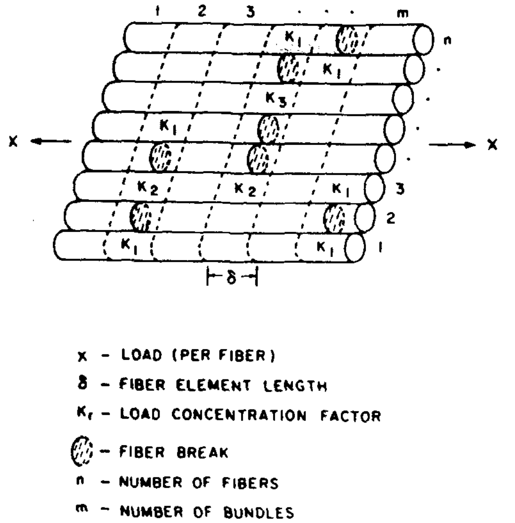

Fig. 1. Fibrous material with structure of a planar tape. The structure is partitioned into m short bundles, each with n fiber elements. Failure is localized within bundles.

and the second is the *power distribution function*

$$F(x) = (x/x_\delta)^\rho, \quad 0 \leqq x \leqq x_\delta \tag{2.2}$$

where ρ and x_δ are the shape and scale parameters respectively in each case. More broadly, we assume $F(x)$ has the form

$$F(x) = G((x/x_\delta)^\rho), \quad x \geqq 0 \tag{2.3}$$

where the function $G(y)$, $y \geqq 0$ is a continuous distribution function with the special properties $G(y) \sim y$ as $y \downarrow 0$ and $1 - G(y) < D\exp(-\alpha y^\beta)$ for $y \geqq 0$, where $\alpha > 0$, $\beta > 0$ and $D > 0$ are constants. (Throughout the paper "$\sim$" will mean that the ratio of the two sides tends to one under the specified condition.) This last property ensures that the upper tail of $F(x)$ is not too long thus avoiding the possibility of too many exceptionally strong fibers. In reality, no fiber can exceed its intrinsic atomic strength, that being bounded. Notice that $G(y) = 1 - e^{-y}$, $y \geqq 0$ in the Weibull case (2.1) and $G(y) = y$, $0 \leqq y \leqq 1$ in the power case (2.2).

A few remarks are in order about these assumptions. The experimental measurement of the strength of fiber elements of length δ is nearly impossible by conventional methods because δ is about 10 fiber diameters in magnitude. The gage length Δ in typical tension tests is about 100 times longer than δ, and by our assumptions such fibers will appear empirically to have a Weibull distribution with shape parameter ρ and lower scale parameter $(\delta/\Delta)^{1/\rho}x_\delta$. In such experiments, the range for the shape parameter ρ is between 5 and 15 (and this is very important in the asymptotic analysis later). The key point is that ρ and the *lower tail* of $F(x)$ for single elements are easily estimated by conventional procedures but only a very small proportion of the test data will be near x_δ. Thus the upper tail of $G(y)$ can be difficult to estimate, and one may have difficulty choosing say between the Weibull form (2.1) and the power form (2.2). We will see later that it is the lower tail of $F(x)$ which is most important, and the precise behavior of $F(x)$ near x_δ is typically of lesser importance.

Bundle geometry and local load-sharing rule

We consider each bundle to be a planar arrangement of parallel elements. If the bundle load is x (per fiber), a surviving fiber element carries load $K_r x$ where

$$K_r = 1 + r/2, \quad r = 1, 2, \ldots, \tag{2.4}$$

and r is the number of consecutive failed fiber elements immediately adjacent to the surviving element (counting on both sides). At the same time a failed fiber element carries no load. We call the K_r *load concentration factors.*

The load sharing rule just described is admittedly an idealization of the true situation in a fibrous material. However, we believe it captures the essential features of the problem.

The bundles described above are referred to as *linear incomplete bundles*; they are linear because the elements form a linear array, and they are incomplete because the load-sharing rule is nonconservative, that is, the total load does not quite sum to nx. The latter difficulty arises when k consecutive broken elements occur at a bundle edge, and no element is available at the outside to support the shed load $(k/2)x$. It turns out that such boundary effects rapidly become unimportant as n increases, and in the interest of simplicity we do not model them.

The basic probability distributions

We let $G_n(x)$, $x \geqq 0$ denote the distribution function for the strength of a *single* bundle, and let $H_{m,n}(x)$, $x \geqq 0$ be the distribution function for the strength of the fibrous material. Since the weakest of the n bundles determines the strength of the structure and the bundles are statistically independent, we have

$$H_{m,n}(x) = 1 - [1 - G_n(x)]^m, \quad x \geqq 0. \tag{2.5}$$

The difficult task is clearly to determine $G_n(x)$, and thus, most of our analysis will focus on a single bundle.

It has proven very fruitful to consider the event of *k-failure* which is the event

$$A_n^{[k]}(x) = \{k \text{ or more failed fiber elements are adjacent somewhere in the bundle under the load } x\};$$

its complement is denoted as $\bar{A}_n^{[k]}(x)$. For this event we define

$$G_n^{[k]}(x) = P\{A_n^{[k]}(x)\}, \quad x \geqq 0. \tag{2.6}$$

Note that $A_n^{[k]}(x)$ is necessary but not sufficient for bundle failure, and thus, $G_n^{[k]}(x)$ is an upper (conservative) bound on $G_n(x)$, $x \geqq 0$. For the fibrous material we let $H_{m,n}^{[k]}(x)$ be the distribution function for k-failure, that is, the probability of occurrence of the event $A_n^{[k]}(x)$ in at least one of its bundles. Thus

$$H_{m,n}^{[k]}(x) = 1 - [1 - G_n^{[k]}(x)]^m, \quad x \geqq 0. \tag{2.7}$$

This concept of k-failure turns out to be very useful because if k is properly selected, then k-failure and bundle failure are virtually equivalent. Under our earlier assumptions on $F(x)$ a fiber element has almost no chance of surviving loads in excess of x_δ. Thus an element flanked by k adjacent broken elements is sure to fail if the bundle load x exceeds x_δ/K_k, and the failure sequence will propagate catastrophically. Such a value of k will be called *critical*, and the sequence of adjacent failed elements, a *critical failure sequence*. (The notion of critical k becomes firm as ρ increases.) Of course, $H_{m,n}^{[k]}(x) \geqq H_{m,n}(x)$ for $x \geqq 0$.

Review of earlier analyses

Harlow and Phoenix[5, 6] were apparently the first to perform rigorous calculations on the exact distribution function for strength $H_{m,n}(x)$ in the setting of composite materials. While their results yielded important insight into the possible significance of k-failure and of the existence of a basic weakest link structure, they found that direct computation was possible only for small bundles. With current computer capacity the practical limit is $n = 12$, and $n = 15$ seems out of the question because of the astronomical growth in the possible failure sequences with n.

The next advance came with the recognition of the importance of k-failure, and Harlow and Phoenix[7–10] developed the recursion analysis (mentioned earlier as Analysis I) for calculating $G_n^{[k]}(x)$ and $H_{m,n}^{[k]}(x)$. The algorithms work in principle for any k, but again the practical limit happens to be $k = 12$. Fortunately $k = 12$ is more than sufficient for practical composites. Harlow and Phoenix considered the simple geometry and local load-sharing rule of this paper. Pitt and Phoenix[11, 12] were able to extend the recursive analysis to certain three-dimensional arrangements for $k = 2$ and more realistic local load-sharing rules in two dimensions, for $k = 3$; in principle the method may be extended to $k > 3$.

While the results of the recursive method are virtually exact, the method itself is cumbersome thus motivating a search for simpler techniques. To this end, Smith[13–15] has developed two asymptotic approaches which we now describe.

The *first* asymptotic approach (Analysis II) fixes k and ρ, and lets $n \to \infty$. This limiting process is similar to that of Analysis I except that no provision is made for evaluating the errors at any fixed n. In fact, a limiting Weibull distribution is obtained which at first sight appears not to be very useful because, in reality, the critical k tends to increase with n. In [14], the approach was extended by allowing k to increase with n but not quite fast enough to keep up with the inherent increase in the critical k.

The second asymptotic approach (Analysis III) is based on the observation that certain key issues arising out of Analysis II may be resolved when ρ is large. In particular the notion of a "critical" value of k becomes firm and moreover, the scale constant arising from the approximation of Analysis II, which in general is quite difficult to compute, may be readily approximated. Based on these ideas, a formal limit theorem was given under the conditions $n \to \infty$ and $\rho \to \infty$ simultaneously such that $(\ln(mn))/\rho \to c$ where c is a positive constant. The idea behind the limit theorem is not that this limiting operation actually takes place in any physical sense, but that an approximation is obtained which may be expected to perform well in applications, where n is typically large and ρ is typically in the range 5–15.

In what follows, our goal is to compare the results of Analysis I of Harlow and Phoenix with those of Analyses II and III of Smith. The resulting accuracy of the latter two asymptotic methods will point to their potential utility in the more general settings.

As a final remark, one might argue that the concept of the ineffective length δ is not realistic because the value itself changes with load x. One may also argue that the concept of the load concentration factor K_i is a gross oversimplification of the truth in view of the actual shape of the stress field on surviving fibers adjacent to breaks. However, Harlow and Phoenix[8] show that these quantities can be defined in a more realistic way, and that the results of the model are not very sensitive to these refinements.

3. SUMMARY OF ANALYSIS I (RECURSION ANALYSIS)

A recursive procedure was developed in[8, 9] to compute $G_n^{[k]}(x)$ for successive values of n when k and x are fixed. In principle, the method allows $G_n^{[k]}(x)$ to be calculated *exactly* for any value of n and for any k for which the required recursion matrix is small enough to be handled by the computer. This recursion matrix grows rapidly as k increases (being of size $2^k - 1$ by $2^k - 1$) so that in practice the method is restricted to $k \leqq 12$.

It was shown that $G_n^{[k]}(x)$ has the structure

$$G_n^{[k]}(x) = 1 - [1 - W^{[k]}(x)]^n[\pi^{[k]}(x) + o_n^{[k]}(x)], \quad x \geqq 0, \quad 1 \leqq k \leqq n, \tag{3.1}$$

where the functions $W^{[k]}(x)$, $\pi^{[k]}(x)$ and $o_n^{[k]}(x)$ are defined as follows:

(1) The function $W^{[k]}(x)$ is the *characteristic distribution function for k-failure.* It is defined as one minus the largest eigenvalue of the recursion matrix, and must be computed numerically given $F(y)$, $y \geqq 0$.

(2) The function $\pi^{[k]}(x)$ is a *boundary term* which reflects load-sharing irregularities at the bundle edges. Under our earlier assumptions on $F(x)$ it turns out that $\pi^{[k]}(x) \to 1$ as $x \to 0$, and $\pi^{[k]}(x)$ does not differ significantly from 1 over all values $x \geqq 0$ of interest. For practical purposes $\pi^{[k]}(x)$ may be taken as 1 except when n is very small.

(3) The function $o_n^{[k]}(x)$, $x \geqq 0$, is a *size residue term.* It may be shown that $o_n^{[k]}(x) \to 0$ as $n \to \infty$ or $x \to 0$, and that it decreases geometrically fast as n is increased. A detailed study of the behavior of $o_n^{[2]}(x)$ is given in [10]. The calculation of $o_n^{[k]}(x)$ for higher k, though possible in principle, is difficult because it is so small. For practical purposes this term may be neglected.

Let

$$\mathscr{H}^{[k]}_{m,n}(x) = 1 - [1 - W^{[k]}(x)]^{mn}, \quad x \geqq 0, \quad 1 \leqq k \leqq n. \tag{3.2}$$

In [9], the following result is proved:

Result 1

Under our earlier assumptions on $F(x)$, $x \geqq 0$,

$$\lim_{n\to\infty} \sup_{x\geqq 0} |H^{[k]}_{m,n}(x) - \mathscr{H}^{[k]}_{m,n}(x)| = 0. \tag{3.3}$$

Moreover, numerical calculations show the error to be negligible for n as small as 10.

The import of these results is that, once k is fixed, the required quantities may be computed very accurately. There remains the question of how to choose k in order to obtain a good approximation to $H_{m,n}(x)$. It appears, however, that from the point of view of practical calculation, this does not cause any difficulties, for the following three reasons:

(1) If there is finite x_{max} such that $F(x_{max}) = 1$, then $H^{[k]}_{m,n}(x) = H_{m,n}(x)$ for all k such that $K_k x > x_{max}$. It is usually the case that this inequality holds for some $k \leqq 12$ so that the foregoing procedures may be applied exactly as described.

(2) Even if there is no such x_{max} but F satisfies (2.3) with ρ not too small, the same procedure, with x_{max} replaced by x_δ, appears to give very good results. This is because (2.3) and the associated assumptions imply that, for most cases of practical interest, $F(y)$ is very close to 1 when $y > x_\delta$, so that the probability of the bundle experiencing k-failure (where $K_k x > x_\delta$, x being the load per fiber) but *not* failing is very small.

(3) It may be proved that the limits

$$W(x) = \lim_{k\to\infty} W^{[k]}(x), \quad \pi(x) = \lim_{k\to\infty} \pi^{[k]}(x) \tag{3.4}$$

exist for each $x \geqq 0$. Moreover, numerical results show that the convergence in (3.4) is very rapid. This leads to the intriguing possibility that a result analogous to (3.1) may hold for $G_n(x)$ with $W(x)$, $\pi(x)$ substituted for $W^{[k]}(x)$, $\pi^{[k]}(x)$ respectively.

A conjecture

Let

$$\mathscr{H}_{m,n}(x) = 1 - [1 - W(x)]^{mn}, \quad x \geqq 0. \tag{3.5}$$

Under our earlier assumptions on $F(x)$ we conjecture that

$$\lim_{n\to\infty} \sup_{x\geqq 0} |H_{m,n}(x) - \mathscr{H}_{m,n}(x)| = 0, \tag{3.6}$$

for fixed m or for any sequence $m \to \infty$. No proof of this conjecture is known but the numerical evidence for it is very strong. It appears that $\rho \geqq 3$ is sufficient for the result to be useful [10].

Result 1 indicates that k-failure in a large fibrous material is essentially a "weakest link" phenomenon where the mn "links" have distribution function $W^{[k]}(x)$, $x \geqq 0$. The conjecture suggests that the same is true for total failure with characteristic distribution function $W(x)$.

To use the above results, one should calculate $W^{[k]}(x)$, $x \geqq 0$, for as large a k as is practicable. Then if n is large

$$H_{m,n}(x) \cong 1 - [1 - W^{[k]}(x)]^{mn} \tag{3.7}$$

will be an accurate and useful approximation over the range $x > x_\delta / K_k$. An example and further comments on applications appear in Section 7.

4. A MOTIVATING EXAMPLE: THREE-FIBER BUNDLES

The later results will depend on certain lower-tail approximations to $G_n^{[k]}(x)$ as $x \to 0$. In order to get an appreciation for these approximations, it is instructive to consider the case $n = 3$, $k = 1, 2, 3$ for which everything can be worked out exactly.

Let X_1, X_2 and X_3 denote the strengths of the three fibers. We need the load concentration factors, K_1 and K_2. The event "bundle fails under load x per fiber" may be decomposed into mutually exclusive events $\{X_1 \leqq x,\ X_2 \leqq x,\ X_3 \leqq x\}$, $\{X_1 \leqq x,\ X_2 \leqq x,\ x < X_3 \leqq K_2 x\}$, $\{x < X_1 \leqq K_1 x,\ X_2 \leqq x,\ K_1 x < X_3 \leqq K_2 x\}$, and so on. Summing the probabilities of these events yields $G_3(x) = F^3(x) + F^2(x)[F(K_2x) - F(x)] + [F(K_1x) - F(x)]F(x)[F(K_2x) - F(K_1x)] + \cdots$ leading to the final result

$$G_3(x) = 4F(x)F(K_1x)F(K_2x) - F(x)F^2(K_1x) - F^2(x)F(K_2x) - 2F^2(x)F(K_1x) + F^3(x), \quad x \geqq 0. \tag{4.1}$$

This is also $G_3^{[3]}(x)$. By similar arguments we find

$$G_3^{[2]}(x) = 4F(x)F(K_1x) - 2F^2(x) - F(x)F^2(K_1x) - 2F^2(x)F(K_1x) + F^2(x)F(K_2x) + F^3(x), \quad x \geqq 0, \tag{4.2}$$

$$G_3^{[1]}(x) = 3F(x) - 3F^2(x) + F^3(x), \quad x \geqq 0. \tag{4.3}$$

In Fig. 2, these distributions have been plotted where F is the Weibull distribution (2.1) or the power distribution (2.2). The values $\rho = 5$ and $\rho = 15$ span the range of practical importance. The scaling is that of Weibull probability paper being linear in $\ln(-\ln(1-p))$ vs $\ln(x)$ where p is cumulative probability. On this scaling the Weibull distribution plots as a straight line. We list several important observations.

Observation 1

Weibull vs power F(x) There is very little difference between the curves for the Weibull and power distributions, especially in the lower-tail region of interest. This is because the two distributions are asymptotically equivalent as $x \to 0$.

Observation 2

Critical loads Transition points in the various distributions appear near $x = x_\delta/K_2$ and $x = x_\delta/K_1$. For example, $G_3(x)$ and $G_3^{[2]}(x)$ are virtually the same for $x > x_\delta/K_2$ and indeed are identical under (2.2). For this reason $k = 2$ is termed the critical failure sequence size when $1/K_2 < x < 1/K_1$.

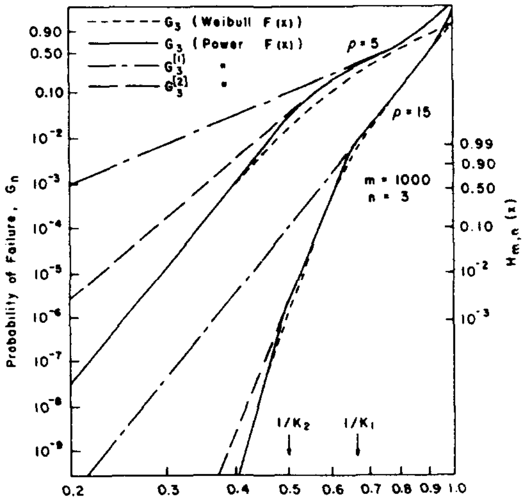

Fig. 2. Distribution functions associated with the failure of a fibrous material with three fibers. Results under the Weibull and power distributions for element strength are compared.

Observation 3

Relation between $H_{m,n}(x)$ *and* $G_n(x)$ The distribution function $H_{m,n}(x)$ is shown for $n = 3$, $m = 1000$. The scale for $H_{m,n}$ is simply that for G_n shifted downward by $\ln(m)$. The fibrous material ($m = 1000$) is much weaker than a single bundle, the reduction in strength being more severe for small ρ. Over the range of interest, $k = 3$ is critical for $\rho = 5$ but $k = 2$ is critical for $\rho = 15$.

Observation 4

The Weibull envelope Notice that each of the three distributions $G_3^{[1]}(x)$, $G_3^{[2]}(x)$, $G_3^{[3]}(x)$ plots to a straight line in the lower tail. If these straight lines are continued to infinity then $G_n(x)$ is very well approximated over its whole range by the lower envelope of the three lines. Each of the three lines corresponds to some Weibull distribution and their lower envelope is called the *Weibull envelope.*

In order to understand Observation 4, we must study eqns (4.1)–(4.3) in more detail. Asymptotically as $x \to 0$, we may replace $F(x)$ by $(x/x_\delta)^\rho$ whereupon (4.1)–(4.3) reduce to

$$\begin{aligned} G_3(x) &= d_3(x/x_\delta)^{3\rho} \\ G_3^{[2]}(x) &= 2d_2(x/x_\delta)^{2\rho} + 0(x^{3\rho}), \\ G_3^{[1]}(x) &= 3d_1(x/x_\delta)^{\rho} + 0(x^{2\rho}), \end{aligned}$$

where

$$d_1 = 1,\ d_2 = 2K_1^\rho - 1,\ d_3 = 4K_1^\rho K_2^\rho - K_1^{2\rho} - K_2^\rho - 2K_1^\rho + 1. \tag{4.4}$$

Note that, apart from the load concentration factors K_1, K_2 (which are taken as given), these constants depend only on ρ, and will be denoted $d_1(\rho)$, $d_2(\rho)$, $d_3(\rho)$ where appropriate.

The result is that we have an approximation

$$G_3^{[k]}(x) \sim (4-k)d_k(\rho)(x/x_\delta)^{k\rho},\ k = 1, 2, 3, \tag{4.5}$$

valid as $x \to 0$. In view of the aforementioned near-equivalence of the Weibull and power distributions, this may also be expressed as

$$\begin{aligned} G_3^{[k]}(x) &\sim 1 - \exp\{-(4-k)d_k(x/x_\delta)^{k\rho}\} \\ &= 1 - \exp\{-(x/x_{4-k}^{[k]})^{k\rho}\},\ k = 1, 2, 3 \end{aligned} \tag{4.6}$$

where

$$x_{4-k}^{[k]} = x_\delta[(4-k)d_k]^{-1/(k\rho)},\ k = 1, 2, 3. \tag{4.7}$$

Thus, we approximate $G_3^{[k]}(x)$ in its lower tail by a Weibull distribution with scale parameter $x_{4-k}^{[k]}$ and shape parameter $k\rho$. Observation 4 amounts to saying that the lower *envelope* of these three Weibull distributions is a good approximation to $G_3(x)$ over almost the whole range of x.

We now consider a different aspect of these approximations. If ρ is large, then both $F(K_2x)/F(K_1x)$ and $F(K_1x)/F(x)$ are much bigger than one as $x \to 0$. This suggests we need retain only the leading terms in (4.1) to (4.3) to yield sufficiently accurate approximations. In the case of the power distribution (2.2), these approximations would be

$$G_3^{[k]}(x) \cong (4-k)\tilde{d}_k(x/x_\delta)^{k\rho},\quad k = 1, 2, 3, \tag{4.8}$$

where

$$\tilde{d}_1 = 1,\ \tilde{d}_2 = 2K_1^\rho,\ \tilde{d}_3 = 4K_1^\rho K_2^\rho. \tag{4.9}$$

In Fig. 3, we have plotted $G_3(x)$ together with the approximations described by (4.8), (4.9), for $\rho = 15$.

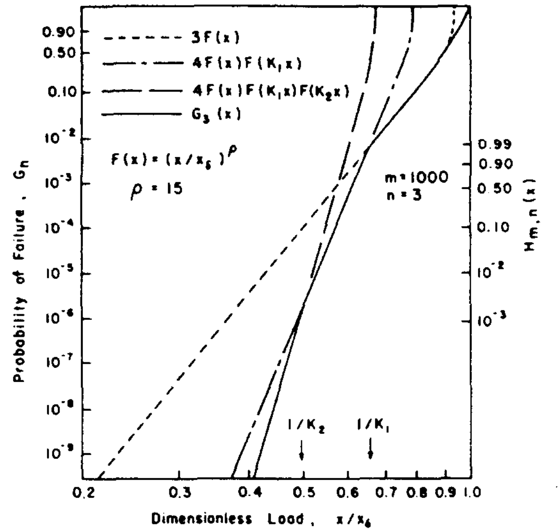

Fig. 3. Leading term approximations to the distribution for bundle and fibrous material strength.

Observation 5

Except in the upper tail, $G_3(x)$ is well approximated by the envelope

$$G_3^*(x) = \min_{1 \leq k \leq 3} (4-k)\tilde{d}_k(x/x_\delta)^{k\rho}, \quad x \geqq 0. \tag{4.10}$$

This approximation is not so good for $\rho = 5$, but a large improvement results in using the Weibull envelope of Observation 4 using (4.6), but with $\tilde{d}_k$ in place of d_k in (4.7). With this replacement, the error is within graphical resolution for $\rho = 5$, and is much smaller for $\rho = 15$.

Summary

These results suggest that a Weibull approximation might be appropriate for $G_n(x)$ and $H_{m,n}(x)$, but that this approximation is dependent on a parameter k which is the critical failure sequence size. For $m = 1000$ and $n = 3$, the critical k is *three* when $\rho = 5$ but only *two* when $\rho = 15$. Furthermore, an approximation to the constant d_k, denoted $\tilde{d}_k$ and based on ρ being large, gives results (in terms of the Weibull approximation) which are nearly indistinguishable even when ρ is as small as 5.

5. OUTLINE OF ANALYSIS II

In the previous section it was shown that the distribution function $G_3^{[k]}(x)$ is well approximated by the Weibull distribution defined by (4.6) and (4.7) in the lower tail. A key result of Analysis II is that a Weibull approximation holds in general for $G_n^{[k]}(x)$ in the lower tail. This result is

$$G_n^{[k]}(x) \sim 1 - \exp\{-(x/x_{n-k+1}^{[k]})^{k\rho}\}, \; k = 1, 2, \ldots, n \tag{5.1}$$

as $x \to 0$ where

$$x_n^{[k]} = x_\delta(nd_k)^{-1/(k\rho)}, \; n = 1, 2, \ldots. \tag{5.2}$$

The constant $d_k = d_k(\rho)$ is given by (4.4) for $k = 1, 2, 3$, and in principle could be computed for any k. In practice it has been computed for k up to 12, the range of practical interest, and some values are given in Table 1. A more detailed table is in[14].

In cases of practical interest we will find that n is very large and k is very small so that it makes no difference if $x_{n-k+1}^{[k]}$ is replaced by $x_n^{[k]}$ in (5.1); in fact, it was in this form that the result was given in[13, 14]. Furthermore, if n is sufficiently large relative to k the Weibull

Table 1. Comparison of the constants d_k of Analysis II and $\tilde{d}_k$ of Analysis III. ($\rho = 5$)

k	d_k	$\tilde{d}_k$	$d_k^{-1/(k\rho)}$	$\tilde{d}_k^{-1/(k\rho)}$
2	1.419 E1	1.519 E1	0.7670	0.7618
3	8.681 E2	9.720 E2	0.6369	0.6322
4	1.565 E5	1.898 E5	0.5499	0.5446
5	6.799 E7	9.226 E7	0.4861	0.4802
6	6.225 E10	9.692 E10	0.4367	0.4303
7	1.088 E14	1.985 E14	0.3971	0.3904
8	3.362 E17	7.325 E17	0.3646	0.3576
9	1.731 E21	4.578 E21	0.3373	0.3301
10	1.414 E25	4.608 E25	0.3140	0.3067
11	1.761 E29	7.167 E29	0.2939	0.2865
12	3.232 E33	1.663 E34	0.2764	0.2689

distribution on the r.h.s. of (5.1) accurately approximates $G_n^{[k]}(x)$ not just in the lower tail but over the whole range of x. Shortly we give conditions for this to be true.

The extension of (5.1) to a fibrous material consisting of m bundles of n fibers is straightforward. The lower tail approximation to $H_{m,n}^{[k]}(x)$ is the r.h.s. of (5.1) with $x_{m(n-k+1)}^{[k]}$ in place of $x_{n-k+1}^{[k]}$. If we let $N = mn$ be the total number of fiber elements in the material, and both N and n are sufficiently large relative to k, then we also have

$$H_{m,n}^{[k]}(x) \simeq 1 - \exp\{-(x/x_N^{[k]})^{k\rho}\} \tag{5.3}$$

over the whole range of $x \geq 0$. In other words, the approximation for a fibrous material is the same as for a single bundle of $N = nm$ fiber elements.

Formally stated, the important result of this section is as follows:

Result 2

Let k be a fixed, positive integer. Then under our earlier assumptions on $F(x)$, $x \geqq 0$ we have

$$\lim_{n\to\infty} |H_{m,n}^{[k]}(x) - [1 - \exp\{-(x/x_N^{[k]})^{k\rho}\}]| = 0. \tag{5.4}$$

(As written this result takes m as fixed, but it also holds even when m grows with n following some sequence $m_1, m_2, \ldots$.)

In the Appendix, we give a sketch of the main ideas underlying the derivation of the above result.

It is interesting to consider the question of how large $N = nm$ must be relative to k for the Weibull approximation (5.3) to be accurate. In [14] it is shown that if k grows with n but at a rate such that N grows at least as fast as $k^{k\phi}$ for some constant $\phi > 1$ depending on F, then (5.3) still holds as an accurate approximation. For the Weibull and power distributions (2.1) and (2.2) for fiber strength, ϕ can be any constant exceeding one. For example, if $k = 5$ we only need N to exceed 3125 but if $k = 10$ we need N exceeding 10^{10}.

As yet we have not considered whether $H_{m,n}(x)$ is accurately approximated by $H_{m,n}^{[k]}(x)$ for some k, and whether (5.3) will apply under these circumstances. This is considered next.

6. OUTLINE OF ANALYSIS III

Although Analysis III may be developed without any reference to Analysis II, as was done in [15], it is most easily understood as an attempt to resolve some of the difficulties posed by Analysis II. Our starting point is the approximation defined by (5.3) and (5.2) using (A4) as the definition for d_k. The two key issues are

(i) the evaluation of d_k

and

(ii) the determination of an appropriate value of k so that (5.3) provides a good approximation not just to $H^{[k]}_{m,n}(x)$ for k-failure in the fibrous material but also to $H_{m,n}(x)$ for total failure.

The analysis hinges on ρ for the fibers being large so that

$$F(x) \cong \begin{cases} (x/x_\delta)^\rho, & 0 < x \leqq x_\delta \\ 1, & x > x_\delta \end{cases}. \tag{6.1}$$

Note that (6.1) is exact under (2.2) but more generally is a good approximation whenever F satisfies (2.3). Thus if some load $x < x_\delta$ is given we have

$$F(K_j x) \cong \begin{cases} K_j^\rho (x/x_\delta)^\rho & \text{if } K_j x < x_\delta \\ 1 & \text{if } K_j x > x_\delta. \end{cases} \tag{6.2}$$

Given load x, define the critical value of k by the inequality

$$K_{k-1}x < x_\delta < K_k x. \tag{6.3}$$

Note that the critical k is not defined when $x = x_\delta/K_k$ for some k. These values of x (which correspond to the transition points on Fig. 2) are excluded from the analysis at present.

From (6.2) we see that $F(K_k x) \cong 1$ so that bundle failure is almost certain as soon as there is a sequence of k consecutive failures. Thus the critical k defined by (6.3) corresponds to the intuitive notion of a critical failure sequence size discussed earlier. Note that, for the time being, our critical k is defined in terms of a given load x rather than the numbers m and n.

We now turn to the evaluation of $G_k(x)$ required for (A4). When ρ is large the variability in fiber strength is small, and it turns out that, in a bundle of size k under load $x < x_\delta/K_{k-1}$, the predominant mode of failure is of a single initial failure under load x followed by the successive failure of neighboring elements under loads $K_1x, K_2x, \ldots, K_{k-1}x$. There are a total of 2^{k-1} ways such a failure progression or crack can grow ([15], p. 544) and using (6.2) we obtain

$$G_k(x) \cong 2^{k-1}(x/x_\delta)^\rho (K_1 x/x_\delta)^\rho \ldots (K_{k-1}x/x_\delta)^\rho. \tag{6.4}$$

This motivates the result that for $x < x_\delta/K_{k-1}$

$$G_k(x) \sim \tilde{d}_k (x/x_\delta)^{k\rho} \tag{6.5}$$

as $\rho \to \infty$ where

$$\tilde{d}_k = \tilde{d}_k(\rho) = 2^{k-1}(K_1 K_2 \ldots K_{k-1})^\rho. \tag{6.6}$$

It can be shown that $\tilde{d}_k$ is an asymptotic approximation to d_k in the sense that

$$\lim_{\rho \to \infty} \tilde{d}_k(\rho)/d_k(\rho) = 1, \tag{6.7}$$

but it turns out that $\tilde{d}_k$ is a very reasonable approximation to d_k even when ρ is as small as 5.

As an illustration we recall the example of Section 4 and notice that (6.4) with $k = 3$ is the leading term of $G_3(x)$, $G_3^{[3]}(x)$ of (4.8) is (6.5), and (4.9) is equivalent to (6.6).

Using $\tilde{d}_k$ in place of d_k we may rewrite (5.1) and (5.2) in the present setting as

$$G_n^{[k]}(x) \sim 1 - \exp\{-(x/\tilde{x}^{[k]}_{n-k+1})^{k\rho}\} \tag{6.8}$$

as $\rho \to \infty$ where

$$\tilde{x}^{[k]}_n = x_\delta [n\tilde{d}_k]^{-1/(k\rho)} \tag{6.9}$$

and k is the critical k defined by (6.3). Note here that "$\sim$" has a different meaning than in

Section 5 in that the condition is now "as $\rho \to \infty$" rather than "as $x \to 0$". However the steps which led to (5.1) also may be modified to yield (6.8) upon noticing that $(x/x_\delta)^\rho \to 0$ and $K_i^\rho/K_j^\rho \to 0$ as $\rho \to \infty$, when $x < x_\delta$ and $i < j$.

Equations (6.8) and (6.9) define the key approximation of Analysis III. This result was given in a different form in [13, 15], but the two forms of the result are equivalent. The details are given in [18].

The extension of these results to a fibrous material is straightforward because the same arguments which lead to (5.3) may be used here also. Thus, we may write

$$H_{m,n}^{[k]}(x) \cong 1 - \exp\{-(x/\bar{x}_N^{[k]})^{k\rho}\} \tag{6.10}$$

where $N = mn$, $n \gg k$, and k and x satisfy (6.3). Furthermore since k is critical, $H_{m,n}^{[k]}(x)$ closely approximates $H_{m,n}(x)$.

Calculation of the critical k

One important question remains to be discussed. So far the critical k has been defined only by (6.3) which relates k to the load x (per fiber) on a bundle or fibrous material. For the case $n = 3$ and $m = 1000$, it was argued in Section 4 that the critical k is three in the case $\rho = 5$ and two in the case $\rho = 15$, and remained constant over the region of interest. The question then is how to relate k to m and n within explicit reference to the load x. We study this only for the fibrous material, since (6.10) obviously reverts to (6.8) when $m = 1$.

The important observation is that, when ρ is large, the probability given by (6.10) is significantly different from both zero and one only if x is very close to $\bar{x}_N^{[k]}$. Thus, for the purpose of determining the critical value of k, we may take $x = \bar{x}_N^{[k]}$ in (6.3). Making this substitution we have

$$x_\delta/K_k < \bar{x}_N^{[k]} < x_\delta/K_{k-1}. \tag{6.11}$$

Next, we substitute for $\bar{x}_N^{[k]}$ from (6.9) using $\bar{d}_k$ from (6.6). Taking logarithms of both sides, we reduce (6.11) to

$$\ln(K_k) > \frac{1}{k\rho}\ln(N) + \frac{k-1}{k\rho}\ln(2) + \frac{1}{k}\sum_{j=1}^{k-1}\ln(K_j) > \ln(K_{k-1}). \tag{6.12}$$

Now suppose N and ρ are both large such that $\ln(N)/\rho \simeq c$ where $0 < c < \infty$. Then (6.12) is approximately

$$k\ln(K_k) > c + \sum_{j=1}^{k-1}\ln(K_j) > k\ln(K_{k-1}). \tag{6.13}$$

This may be rewritten as

$$\gamma(k) > c > \gamma(k-1) \tag{6.14}$$

where $\gamma(0) = 0$ and $\gamma(r)$ is defined for the positive integers by

$$\gamma(r) = r\ln(K_r) - \sum_{j=1}^{r-1}\ln(K_j),\quad r = 1, 2, \ldots. \tag{6.15}$$

Note that because the sequence $\{K_r, r \geqq 1\}$ is strictly increasing, the same is true of the sequence $\{\gamma(r), r \geqq 1\}$.

Thus we conclude that, for large N and ρ, and $c = (\ln N)/\rho$, the critical value of k may be defined by (6.14) and (6.15), and this value used in (6.10). Henceforth, we refer to this critical value of k as k^*.

A small problem arises if $\gamma(r) = c$ for some r. In this case k^* may be defined to be either r

or $r+1$. It may be checked that the two corresponding values of $\bar{x}_N^{[k^*]}$ are almost the same, so it seems to make little difference in practice.

A formal statement of the main result is as follows:

Result 3

Suppose ρ grows with n following a sequence $\rho, \rho_2, \ldots$ such that $\ln(N)/\rho_n \to c$ as $n \to \infty$ where $N = mn$ and c is a positive constant. Then if F satisfies (2.3) we have

$$\lim_{n\to\infty} \sup_{x\geq 0} |H_{m,n}(x) - [1 - \exp\{-(x/x_N^{[k^*]})^{k^*\rho_n}\}]| = 0, \tag{6.16}$$

where k^* is the value of k solving (6.14). (As written this result takes m as fixed, but it also holds even when m grows with n following some sequence $m_1, m_2, \ldots$.)

To summarize, the key approximation is

$$H_{m,n}(x) \cong 1 - \exp\{-(x/\bar{x}_N^{[k^*]})^{k^*\rho}\}, \quad x \geqq 0 \tag{6.17}$$

where $\bar{x}_N^{[k^*]}$ is given by (6.9) with (6.6), and k^* is the critical value of k which solves (6.14) and (6.15) with $c = \ln(N)/\rho$.

For convenience we list the first few values of $\gamma(r)$ when $K_j = 1 + j/2$:

r	$\gamma(r)$	r	$\gamma(r)$
0	0	5	3.15
1	0.405	6	3.95
2	0.981	7	4.78
3	1.65	8	5.62
4	2.38	9	6.48

To conclude this section we mention one other result of [15] which is less important for applications but may be of independent interest. Consider a formal limiting operation as $\rho \to \infty$, $N \to \infty$, $(\ln N)/\rho \to c$. Then the critical k^* is the value of k defined by (6.14) and it may be shown with a little algebra that

$$\bar{x}_N^{[k^*]}/x_\delta \to \exp\left[-\left(c + \sum_{j=1}^{k^*-1} \ln(K_j)\right)/k^*\right]. \tag{6.18}$$

Note that the r.h.s. of (6.18) depends only on c and may be written $\mu(c)$. Furthermore, the Weibull shape parameter ρk^* in (6.17) tends to infinity so that the variance of the distribution tends to zero. Thus, as $N \to \infty$ and $\rho \to \infty$ such that $(\ln N)/\rho \to c$, the strength of the material approaches a limiting value which may be written as $x_\delta \mu(c)$.

Of course, the limiting operation described here never takes place physically but the result does suggest that, when N and ρ are both large, the strength of the material is close to $x_\delta \mu((\ln N)/\rho)$. It now seems that, as a working approximation for strength, this result is of less importance than (6.17), but it is still important for the light it casts on the "size effect." The reader is referred to [15] for discussion of this concept. A later example illustrates this result.

7. COMPARISON OF THE THREE ANALYSIS AND CONCLUDING COMMENTS

Analysis I is the most accurate of the three analyses, since it leads to very close approximations to $H_{m,n}^{[k]}(x)$ with tight bounds on the error of approximation. The problem of choosing k does not really arise, because the sequence of approximations (as k incregses) can be shown theoretically to converge to a limit, and numerically it is known that this limit is approached very fast. For large values of n, the approximation is given by (3.7) or in a virtually equivalent form by

$$H_{m,n}(x) \cong 1 - [1 - W(x)]^{mn}, \quad x \geqq 0, \tag{7.1}$$

where $W(x)$ is defined as the limit of $W^{[k]}(x)$ as k increases.

The key result of Analysis II is (5.3) which (writing $N = mn$) we rewrite here in the form

$$H_{m,n}(x) \cong 1 - \exp\{-mnd_k(x/x_\delta)^{k\rho}\}, \quad x \geqq 0 \tag{7.2}$$

where the value of k must be chosen as indicated shortly. Comparing (7.2) with (7.1), it is seen that this is equivalent to replacing $W(x)$ by the Weibull distribution function

$$\mathscr{F}_1^{[k]}(x) = 1 - \exp\{-d_k(x/x_\delta)^{k\rho}\}, \quad x \geqq 0. \tag{7.3}$$

The problem of choosing k in (7.2) is much more subtle. Formula (7.2) is based on a lower tail approximation, and it was seen in Section 5 that the rate of approach to the limit as n grows large gets slower as k increases. The practical conclusion which we must note here is that if k is chosen too large then (7.2) fails. Of course, if k is chosen too small then (7.2) also fails simply because $H_{m,n}^{[k]}(x)$ is not then a good approximation to $H_{m,n}(x)$. This makes it clear that it is necessary to choose the right value of k if there is any chance for (7.2) to be a good approximation.

Two practical approaches have been suggested for resolving this question. The first is to use the critical value of k given by Analysis III. This seems a very reasonable thing to do. The second approach is motivated by the observation that, in the circumstances when (7.2) fails, it always does so in the direction of *overestimating* the true probability of failure. This motivates the Weibull envelope procedure, introduced in Section 4, whereby $\mathscr{F}_1^{[k]}(x)$ from (7.3) is replaced by

$$\hat{W}(x) = \min_{k \geq 1} \mathscr{F}_1^{[k]}(x), \quad x \geqq 0. \tag{7.4}$$

The resulting approximation is then

$$H_{m,n}(x) \cong 1 - [1 - \hat{W}(x)]^{mn}, \quad x \geq 0. \tag{7.5}$$

Analysis III produces two results worth studying. The first is the approximation (6.17) which (writing $N = mn$) we rewrite here as

$$H_{m,n}(x) \cong 1 - \exp\{-mn\tilde{d}_{k^*}(x/x_\delta)^{k^*\rho}\}, \quad x \geq 0 \tag{7.6}$$

with the critical k^* being the value of k solving (6.14) and (6.15) with $c = (\ln(mn))/\rho$. The second result is that when mn and ρ are both large, the strength of the material is approximately x_δ times the r.h.s. of (6.18), which we denote $\mu(c)$, with $c = (\ln(mn))/\rho$.

A comparison of these approximations is made in Fig. 4. We have plotted the charac-

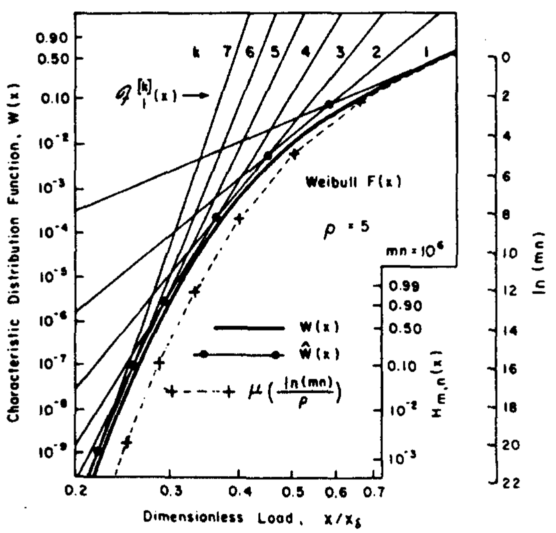

Fig. 4. Characteristic distribution function $W(x)$ for material strength and associated approximations. Also shown is the strength distribution for a fibrous material with 10^6 fiber elements

teristic distribution function $W(x)$ and the approximating functions $\mathcal{F}_1^{[k]}(x)$ on Weibull coordinates. Also shown (right-hand scale) are the estimates (7.1) and (7.2) for the case $mn = 10^6$. These curves are obtained from $W(x)$ and $\mathcal{F}_1^{[k]}(x)$ respectively by shifting the left-hand vertical scale downward an amount $\ln(mn) = 13.82$.

It is apparent from Fig. 4 that there is no single value of k for which $\mathcal{F}_1^{[k]}(x)$ provides a good approximation to $W(x)$ over the whole range of x. It may be seen, however, that $\mathcal{F}_1^{[k]}(x)$ is nearly tangential to $W(x)$ over the region where k is critical, and that $\hat{W}(x)$ defined by (7.4) provides a very reasonable approximation to $W(x)$ over almost the whole range. For the case $mn = 10^6$, the median value of x ($H_{m,n}(x) = 0.5$) lies in the region where $\hat{W}(x) = \mathcal{F}_1^{[5]}(x)$, and we conclude that $k = 5$ is critical for this mn and $\rho = 5$.

Thus it appears that the Weibull approximation defined by (7.2) with $k = 5$ is a good approximation for $mn = 10^6$, $\rho = 5$. This distribution has a shape parameter of 25 as compared with 5 for a single fiber so that the variability in strength of the fibrous material is greatly reduced in comparison with single fibers. This is an important observation in itself which is consistent with experimental evidence. (A study of past and recent experimental evidence is to be published elsewhere.)

A table of values for d_k, needed for Analysis II is given in Table 1 for the case $\rho = 5$. A more detailed table, covering different values of ρ, is in [14]. Numerical comparisons of $W(x)$ with the envelope estimate $\hat{W}(x)$ of (7.4) show that the approximation is a good one and also that $\hat{W}(x)$ appears to *overestimate* $W(x)$, so that the approximation of Analysis II is *conservative.*

We now turn to a comparison of Analyses II and III. The main approximation of Analysis III is (7.6) which (disregarding the choice of $k = k^*$ for the moment) is the same as Analysis II with $\bar{d}_k$ replacing d_k. In Table 1 we have given values of d_k and $\bar{d}_k$ and also the values of $d_k^{-1/(k\rho)}$ and $\bar{d}_k^{-1/(k\rho)}$. The latter values are tabulated because the scale constants defined in (5.2) and (6.9) are proportional to them. It may be seen that the third and fourth columns are very close. In fact, the error that arises on Fig. 4, when $\bar{d}_k$ is replaced by d_k in (7.3), is within graphical resolution.

The other issue of Analysis III is the choice of k. However, for $mn = 10^6$, $\rho = 5$ we find $(\ln(mn))/\rho = 2.763$ and $\gamma(4) = 2.38 < 2.763 < \gamma(5) = 3.15$. Thus the critical value of k as determined by Analysis III is $k^* = 5$, i.e. the same value as was found graphically from Fig. 4. This appears to be true generally so that the predictions of Analyses II and III appear to be the same for all practical purposes. This is fortunate because Analysis III was derived on the assumption of $\rho \to \infty$ but the preceeding conclusions imply that the assumptions of Analysis III are justified even for $\rho = 5$.

The other prediction of Analysis III is that the strength is approximately $x_\delta \mu(\ln(mn)/\rho)$. This curve is also plotted in Fig. 4 using the $\ln(mn)$ scale on the extreme right. Although the approximation is still a reasonable one, it is not as good as the others and apparently overestimates the true material strength, i.e. the approximation is non-conservative.

Boundary effects when n *is small*

The approximations (7.1), (7.2) and (7.3) are valid whenever n is large regardless of the value of m. When n is small but m large the approximations are still good but some modification is needed to account for boundary effects in the small bundles. These boundary effects arise because the mathematical analysis does not take account of physical changes near the boundary which become appreciable when $(n - k^* + 1)/n$ is significantly less than one.

When using Analysis I, numerical experience indicates that formula (7.1) should be replaced by

$$H_{m,n}(x) \cong 1 - (1 - W(x))^{mn}(\pi(x))^m, \quad x \geqq 0 \tag{7.7}$$

say for $k^* \leqq n \leqq 3k^*$, where k^* is the critical k of Analysis III and $\pi(x)$ is given by (3.4). For even smaller values of n, i.e. $n < k^*$ one should use

$$H_{m,n}(x) \cong 1 - [1 - W^{[n]}(x)]^{mn}(\pi^{[n]}(x))^m, \quad x \geqq 0, \tag{7.8}$$

where $\pi^{[n]}(x)$ is defined in association with (3.1). In fact, (7.8) also actually works better than (7.7) even for $n \geqq k^*$ especially in the lower tail of $H_{m,n}(x)$.

When using Analysis II, the corresponding change in (7.2) is

$$H_{m,n}(x) \cong 1-\exp\{-m(n-k+1)d_k(x/x_\delta)^{k\rho}\}$$
$$= 1-[1-\mathcal{F}_1^{[k]}(x)]^{mn}[1-\mathcal{F}_1^{[k]}(x)]^{-m(k-1)}, \ x \geqq 0 \tag{7.9}$$

where k may be taken as k^* of Analysis III, when $k^* \leqq n \leqq 3k^*$, or as n when $n < k^*$. A comparison of (7.9) with (7.7) in view of (7.5) suggests a "Weibull envelope" approximation

$$H_{m,n}(x) \cong 1-[1-\hat{W}(x)]^{mn}(\hat{\pi}(x))^m, \ x \geqq 0 \tag{7.10}$$

where $\hat{W}(x)$ was defined earlier by (7.4) and where

$$\hat{\pi}(x) = \min_{k\geq 1} \exp\{(k-1)d_k(x/x_\delta)^{k\rho}\}, \ x \geqq 0. \tag{7.11}$$

(Actually (7.10) differs slightly from (7.9) but only at a few points x where it is still within graphical resolution on Weibull probability paper.)

Numerical calculations have been performed which show that $\pi(x)-1$ and its approximation $\hat{\pi}(x)-1$ are very close numerically especially in the load range x of interest. The proximity is roughly the same as that shown on Fig. 4 for $W(x)$ and its approximation $\hat{W}(x)$, and it improves dramatically as ρ increases. For $x_\delta/K_k < x < x_\delta/K_{k-1}$ we see that $(\pi(x))^m - 1 \cong m(k-1)d_k(x/x_\delta)^{k\rho}$ so that $(\pi(x))^m$ becomes significant in (7.10) only when $m(k-1)$ approaches mn in magnitude.

Concluding comments

For parameter values in the range of interest the three analyses yield results which are in remarkable agreement. The Weibull approximation (7.6), using (6.6) for d_k and either of the suggested rules for determining k^*, yields very good results in spite of the fact that very little labor is involved in calculating the key constants.

Future work will concentrate on extending these results to fatigue failure and different bundle geometries and load-sharing rules. The favorable comparisons made in this paper are important because at the present time only Analyses II and III show real prospects of being extended to these situations. Analysis II has been extended to fatigue failure in[19], and work is in progress on extending Analyses II and III to bundles with hexagonal geometry under local load-sharing.

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APPENDIX

Consider a single bundle and suppose the n fiber elements are labeled consecutively $i = 1, 2, \ldots, n$. The event of k-failure requires that there is some group of k consecutive fibers which fails under the given load. In order to obtain the distribution for this, it is convenient to consider fibers $i, i+1, \ldots, i+k-1$ (for some i between 1 and $n-k+1$) as an *isolated* bundle of k fibers. The probability of failure of this small bundle is simply $G_k(x)$. In general, we may define the event $A_{n,i}$ for $n \geq 1$, $1 \leqq i \leqq n-k+1$ as the event that fibers $i, i+1, \ldots, i+k-1$, in a bundle of size n, fail when considered in this manner. Note that $A_{n,i}$ really depends on both k and the applied load x, but this is suppressed in the notation. We then have the approximation

$$G_n^{[k]}(x) \cong P\{\bigcup_i A_{n,i}\} \tag{A1}$$

where the union of the $A_{n,i}$ is taken over $1 \leqq i \leqq n-k+1$. The reader may be asking why (A1) is only an approximation and not an exact result. This question is important because it must be answered in order to see why (A1) really is a good approximation. The point is that k-failure may occur without any of the events $A_{n,i}$ occurring, through the linking-up of two sequences of failures to produce a single sequence which is actually longer than k.

To make the point clear, consider $k = 2$. Suppose fibers 1, 2, 3 have strengths X_i satisfying

$$X_1 \leqq x, \quad K_1 x < X_2 \leqq K_2 x, \quad X_3 \leqq x.$$

Fibers 1 and 2 do not fail when considered as a bundle of only two fibers because fiber 2 can withstand the overload from fiber 1. Hence $A_{n,1}$ does not occur. Similarly $A_{n,2}$ does not occur. But it is clear that, when the three fibers are considered together, they all fail because fiber 2 fails under the combined overloads from fibers 1 and 3. In this case, we have 2-failure occurring, but only through the combined weaknesses of *three* fibers.

In general, we use the term "Type II failure sequences" for a sequence of consecutive fibers, of length *strictly* greater than k, which fail under their combined loads but such that no $A_{n,i}$ occurs within the sequence. If $A'_{n,i}$ denotes the event that fibers $i, i+1, \ldots, i+k-1$ are contained within some Type II failure sequence, then (A1) may be modified to give the equality

$$G_n^{[k]}(x) = P\{\bigcup_i (A_{n,i} \cup A'_{n,i})\}, \quad x \geqq 0. \tag{A2}$$

In view of the inequalities

$$P\{\bigcup_i A_{n,i}\} \leqq G_n^{[k]}(x) \leqq P\{\bigcup_i A_{n,i}\} + P\{\bigcup_i A'_{n,i}\} \tag{A3}$$

it will suffice, to justify (A1), to show that the events $A'_{n,i}$ have negligible probability in comparison with the events $A_{n,i}$ as $x \to 0$. We return to this point shortly.

The next problem is to obtain an approximation to the probability of $A_{n,i}$, i.e., to $G_k(x)$. We showed in Section 4 that as $x \to 0$ the approximation

$$G_3(x) \sim d_3(x/x_\delta)^{3\rho}$$

holds. By similar arguments for $G_1(x) = F(x)$ and $G_2(x) = 2F(K_1 x)F(x) - F^2(x)$ we see under the previously stated assumptions on F that

$$P\{A_{n,i}\} = G_k(x) \sim d_k(x/x_\delta)^{k\rho}, \quad x \to 0 \tag{A4}$$

holds for $k = 1, 2, 3$. We claim that, for *each* $k \geqq 1$, there exists $d_k = d_k(\rho)$ such that (A4) holds. An intuitive "proof" of this is as follows. We found that $G_3(x)$ is a sum of products of *three* factors of the form $F(vx) - F(ux)$ where v and u are load concentration factors (possibly 1 or 0). In general, it is easy to visualize how this argument may be extended to obtain $G_k(x)$ as a sum of products of k such factors. Using the approximation $F(y) \sim (y/x_\delta)^\rho$ as $y \to 0$ we obtain (A4). A rigorous proof of (A4) by the method of induction is given in [16].

Equation (A4) gives an approximation for $P\{A_{n,i}\}$ which is valid when the applied load x is small. The next step is to apply this to obtain the probability of the union of the events $A_{n,i}$. We approach this problem by first making the artificial (and incorrect) assumption that the events $A_{n,i}$ (for $i = 1, \ldots, n-k+1$) are independent. Under this assumption we have

$$P\{\bigcup_i A_{n,i}\} = 1 - P\{\bigcap_i \bar{A}_{n,i}\} = 1 - \prod_i P\{\bar{A}_{n,i}\}$$
$$= 1 - [1 - G_k(x)]^{n-k+1}$$

where $\bar{A}_{n,i}$ denotes the complement of $A_{n,i}$. Now in the region of interest x is very small and so is $G_k(x)$. Thus we have

$$1 - G_k(x) \sim \exp\{-G_k(x)\} \sim \exp\{-d_k(x/x_\delta)^{k\rho}\}$$

using (A4). This yields the approximation

$$P\{\bigcup_i A_{n,i}\} \sim 1-\exp\{-(n-k+1)d_k(x/x_\delta)^{k\rho}\}$$
$$= 1-\exp\{-(x/x^{[k]}_{n-k+1})^{k\rho}\} \text{ as } x \to 0 \quad \text{(A5)}$$

where $x^{[k]}_{n-k+1}$ is defined by (5.2).

Now (A5) is identical to (5.1) but it was derived under the incorrect assumption that the $A_{n,i}$'s are independent. However, it is possible to argue that, in the limit as $x \to 0$, the events $\{A_{n,i}, 1 \leqq i \leqq n-k+1\}$ are *asymptotically* independent. The argument depends on a theorem by Watson[17], and to use the theorem one must show that

$$P\{A_{n,j}|A_{n,i}\} \to 0 \text{ as } x \to 0 \quad \text{(A6)}$$

for $j \neq i$. The difficulty arises when $|j-i|<k$ because then the events $A_{n,j}$ and $A_{n,i}$ overlap in the bundle, and are dependent because they share common fibers. Now

$$P\{A_{n,j}|A_{n,i}\} = P\{A_{n,j} \cap A_{n,i}\}/P\{A_{n,i}\} \quad \text{(A7)}$$

where $|j-i|<k$, and the event $A_{n,j} \cap A_{n,i}$ requires that *at least* $k+1$ fibers fail under loads which can at most be $K_{k-1}x$. Thus by arguments similar to those which led to (A4) we have

$$P\{A_{n,j} \cap A_{n,i}\} < d'_k(x/x_\delta)^{(k+1)\rho} \quad (8)$$

for all $0 \leqq x < x_k$ where d'_k and x_k are positive constants whose precise values need not concern us. From (A8) and (A4) we arrive at

$$0 < P\{A_{n,j}|A_{n,i}\} < (d'_k/d_k)(x/x_\delta)^\rho$$

for all x sufficiently small, thus verifying (A6). We see that even if fibers $i, i+1, \ldots, i+k-1$ are *known* to have failed under the event $A_{n,i}$, the chance that fibers $i+1, i+2, \ldots, i+k$ have also failed by the neighboring event $A_{n,i+1}$ is negligible when the load x is small in spite of the fact that $k-1$ of the fibers are common to both events!

We now return to the issue of Type II failure sequences and the events $A'_{n,i}$. Now each $A'_{n,i}$ $(i = 1, 2, \ldots, n-k+1)$ requires the failure of *at* least $k+1$ adjacent fibers. By arguments similar to those which led to (A4), we have as $x \to 0$ the approximation

$$P\{A'_{n,i}\} \sim d''_k(x/x_\delta)^{(k+1)\rho}, \quad x \geqq 0$$

for some constant d''_k whose precise value need not concern us. Combined with (A4), We have

$$P\{A'_{n,i}\} \sim (d''_k/d_k)(x/x_\delta)^\rho P\{A_{n,i}\}.$$

Now d_k and d''_k are constants while $(x/x_\delta)^\rho \to 0$ as $x \to 0$. Thus, as $x \to 0$, the events $A'_{n,i}$ are of negligible probability compared with the events $A_{n,i}$. An easy extension of this argument (combined with (A5)) shows that $P\{\bigcup_i A'_{n,i}\}$ is negligible compared with $P\{\bigcup_i A_{n,i}\}$ as $x \to 0$. In view of (A3) this shows that the approximation (A1) is justified when x is small. Finally (A1) together with (A5) justify the approximation (5.1).

Lastly, we consider the reasons why the Weibull distribution on the r.h.s. of (5.1) improves as an approximation to $G^{[k]}_n(x)$ as n grows large and k remains fixed. The key is to return to the derivation of (A5) and notice that $[1-z/n]^n \to e^{-z}$ as $n \to \infty$ for any real z so that the exponential form arises naturally as n grows large. Second, the median of this Weibull distribution is $x_\delta[d_k(n-k+1)/\ln(2)]^{-1/k\rho}$ and decreases as $n \to \infty$ in proportion to $n^{-1/(k\rho)}$. Thus as n grows large the load range of interest for x moves into the region where the lower tails for the probabilities of the events $A_{n,i}$ and $A'_{n,i}$ dominate in importance.

The above ideas apply to a single bundle when n grows large with k held fixed and are developed more rigorously in[13, 14]. These ideas are easily extended to the fibrous material where there are a total of $2m(n-k+1)$ events to consider, each of the form $A_{n,i}$ or $A'_{n,i}$. Then $H^{[k]}_{m,n}(x)$ still has the form of the r.h.s. of (A2) but the union is over $m(n-k+1)$ such events. Furthermore, events in different bundles are automatically independent thus simplifying the analysis, the final result of which is (5.3) and Result 2.